Conditioning on the margins of a 2×2 table

Re-parameterizing under the multinomial model yields a modest simplification in the algebra, but in the product binomial model, just plowing ahead with the original parameters is simplest.

$$
\begin{aligned}
Pr\{a,b,c,d\} &= \binom{n_1}{a}\binom{n_2}{b}\pi_1^a(1-\pi_1)^{n_1-a}\pi_2^b(1-\pi_2)^{n_2-b} \\
& \quad re-expressing\ b,\ c,\ and\ d\ in\ terms\ of\ n_1\ and\ m_1\ yields \\
&= \binom{n_1}{a}\binom{n_2}{m_1-a}\pi_1^a(1-\pi_1)^{n_1-a}\pi_2^{m_1-a}(1-\pi_2)^{n_2-m_1+a} \\
& \quad factoring\ out\ terms\ involving\ a\ results\ in \\
&= \binom{n_1}{a}\binom{n_2}{m_1-a}\left(\frac{\pi_1/(1-\pi_1)}{\pi_2/(1-\pi_2)}\right)^a(1-\pi_1)^{n_1}\pi_2^{m_1}(1-\pi_2)^{N-(n_1+m_1)}
\end{aligned}
$$

To get the conditional probability of the observed value a, given the margins n_1, m_1 we need to divide this by the marginal distribution of n_1 and m_1, which is obtained by summing the above for all possible values of a. Noting that the last three terms are constant in this summation, they cancel out in the conditional probability, yielding the desired expression.